Definition 7.5:

For a function $f: D \longrightarrow \mathbb{R}$ define functions f_+, $f_-: D \longrightarrow \mathbb{R}$ as follows:

$$f_+(x) := \begin{cases} f(x), & \text{if } f(x) > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$f_-(x) := \begin{cases} -f(x), & \text{if } f(x) < 0, \\ 0 & \text{otherwise.} \end{cases}$$

Apparently we have $f = f_+ - f_-$ and $|f| = f_+ + f_-$.

Proposition 7.6:

Let $f, g: [a,b] \longrightarrow \mathbb{R}$ be integrable functions.
Then we have:

i) The functions f_+, f_- and $|f|$ are integrable and we have

$$\left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx.$$

ii) For every $p \in [1, \infty)$ the function $|f|^p$ is integrable.

iii) The functions $f \cdot g: [a,b] \longrightarrow \mathbb{R}$ is integrable.

Proof:

i) According to assumption, for given $\varepsilon > 0$ there exist step functions $\varphi, \psi \in S[a,b]$ with $\varphi \leq f \leq \psi$ and

$$\int_a^b (\psi - \varphi)(x)\,dx \leq \varepsilon .$$

Then φ_+ and ψ_+ are also step functions with $\varphi_+ \leq f_+ \leq \psi_+$ and

$$\int_a^b (\psi_+ - \varphi_+)(x)\,dx \leq \int_a^b (\psi - \varphi)(x)\,dx \leq \varepsilon ,$$

therefore f_+ is integrable. Analogously, f_- is integrable as well. According to Prop. 7.5 $|f|$ is then also integrable. As $f \leq |f|$ and $-f \leq |f|$, Prop. 7.5 iii) gives

$$\int_a^b f(x)\,dx \leq \int_a^b |f(x)|\,dx .$$

ii) It suffices to show integrability of $|f|^p$ in the case $0 \leq f \leq 1$. For $\varepsilon > 0$ there are step functions $\varphi, \psi \in S[a,b]$ with

$$0 \leq \varphi \leq f \leq \psi \leq 1$$

and
$$\int_a^b (\psi - \varphi)\,dx \le \frac{\varepsilon}{p}.$$

$\Rightarrow$ φ^p and ψ^p are also step functions with $\varphi^p \le f^p \le \psi^p$ and due to
$$\frac{d}{dx}(x^p) = p x^{p-1}$$
the meanvalue theorem gives
$$\psi^p - \varphi^p \le p(\psi - \varphi).$$

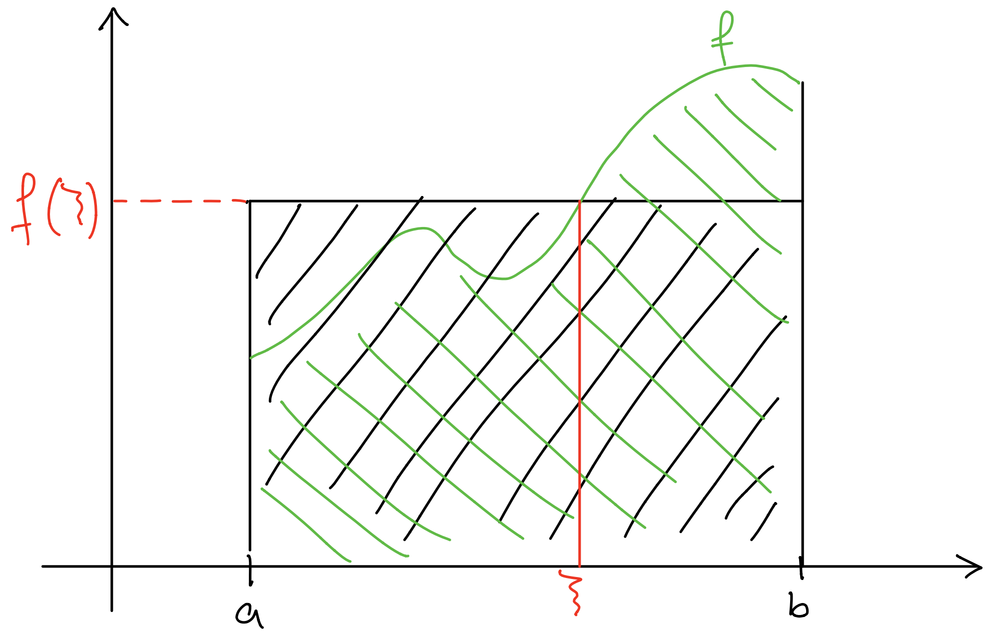
Therefore,
$$\int_a^b (\psi^p - \varphi^p)(x)\,dx \le p \int_a^b (\psi - \varphi)(x)\,dx \le \varepsilon,$$
thus f^p is integrable.

iii) The claim follows from
$$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right].$$

<u>Proposition 7.7</u> (Mean value theorem of integration):
Let $\varphi : [a,b] \longrightarrow \mathbb{R}_+$ be a non-negative integrable function. Then there exist for every continuous function $f : [a,b] \longrightarrow \mathbb{R}$ a $\xi \in [a,b]$, s.t.
$$\int_a^b f(x)\,\varphi(x)\,dx = f(\xi) \int_a^b \varphi(x)\,dx.$$

In the special case $\varphi = 1$ one gets

$$\int_a^b f(x)\,dx = f(\xi)(b-a) \quad \text{for } \xi \in [a,b].$$



For an arbitrary integrable function $f : [a,b] \to \mathbb{R}$ one calls

$$M(f) := \frac{1}{b-a} \int_a^b f(x)\,dx$$

the mean value of f over the interval $[a,b]$. More generally, we call

$$M_\varphi(f) := \frac{1}{\int_a^b \varphi(x)\,dx} \int_a^b f(x)\,\varphi(x)\,dx$$

the "weighted mean value" of f (if $\int_a^b \varphi(x)\,dx \neq 0$).

Proof:

According to Prop. 7.6 the function $f\varphi$ is integrable. We set

$$m := \inf\{ f(x) \mid x \in [a,b]\},$$
$$M := \sup\{ f(x) \mid x \in [a,b]\}.$$

Then we have $m\varphi \leq f\varphi \leq M\varphi$, so according to Prop. 7.5:

$$m \int_a^b \varphi(x)\,dx \leq \int_a^b f(x)\varphi(x)\,dx \leq M \int_a^b \varphi(x)\,dx.$$

Therefore, there exists a number $\mu \in [m, M]$ s.t.

$$\int_a^b f(x)\,\varphi(x)\,dx = \mu \int_a^b \varphi(x)\,dx.$$

Int. value theorem $\Rightarrow \exists\, \xi \in [a,b]$ s.t. $f(\xi) = \mu$. $\Rightarrow$ claim follows. $\square$

Recall:

Theorem 7.1:

Let $f:[a,b] \longrightarrow \mathbb{R}$ be a Riemann-integrable function. There there exists for every $\varepsilon>0$ a $\delta>0$, such that for every choice Z of points x_k and ξ_k of fineness $\mu(Z)<\delta$ we have:

$$\left| \int_a^b f(x)\,dx - S(Z,f) \right| \leq \varepsilon .$$

One can also write this as follows:

$$\lim_{\mu(Z)\to 0} S(Z,f) = \int_a^b f(x)\,dx.$$

Proof:

Let φ, ψ be step functions with $\varphi \leq f \leq \psi$

$\Rightarrow$ For all sub-divisions Z:

$$S(Z,\varphi) \leq S(Z,f) \leq S(Z,\psi)$$

Thus it suffices to prove the claim in the case where f is a step function.

Choose the sub-division

$$a = t_0 < t_1 < \cdots < t_m = b$$

As f is bounded, there exists
$$M := \sup\{|f(x)| \mid x \in [a,b]\} \in \mathbb{R}_{(>0)}.$$
Let $\mathcal{Z} := ((x_k)_{0\leq k\leq n}, (\xi_k)_{1\leq k\leq n})$ some sub-division of the interval $[a,b]$ and $F \in S[a,b]$ the step function defined by
$$F(a) = f(a) \text{ and } F(x) = f(\xi_k) \text{ for } x_{k-1} < x \leq x_k.$$
Then we have
$$S(\mathcal{Z}, f) = \int_a^b F(x)\,dx,$$
therefore
$$\left|\int_a^b f(x)\,dx - S(\mathcal{Z}, f)\right| \leq \int_a^b |f(x) - F(x)|\,dx.$$
The functions f and F are equal on all sub-intervals (x_{k-1}, x_k) with $t_j \notin [x_{k-1}, x_k]\ \forall j$.
$\Rightarrow$ different at most on $2m$ sub-intervals (x_{k-1}, x_k) of total length $2m\mu(\mathcal{Z})$. Moreover,
$$|f(x) - F(x)| \leq 2M,$$
thus
$$\int_a^b |f(x) - F(x)|\,dx \leq 4mM\mu(\mathcal{Z})$$

As $\mu(Z) \to 0$, the claim follows. $\square$

Example 7.3:

We want to compute the integral $\int_0^a \cos x \, dx$, $(a > 0)$, using Riemann sums. For a natural number $n \in \mathbb{N}$, we set:

$$x_k := \frac{ka}{n}, \quad k = 0, 1, \dots, n,$$

giving an equidistant sub-division of the interval $[0, a]$ of fineness $\frac{a}{n}$. As support points we choose $\xi_k = x_k$. The corresponding Riemann sum is

$$S_n = \sum_{k=1}^{n} \frac{a}{n} \cos \frac{ka}{n} \qquad (*)$$

Now use the following:

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{\sin\left(n + \frac{1}{2}\right)t}{2 \sin \frac{1}{2} t} \quad \text{for } t \notin 2\pi\mathbb{Z}$$

Proof:

We have $\cos kt = \frac{1}{2}\left(e^{ikt} + e^{-ikt}\right)$, thus

$$\frac{1}{2} + \sum_{k=1}^{n} \cos kt = \frac{1}{2} \sum_{k=-n}^{n} e^{ikt}.$$

On the other hand, we have

$$\sum_{k=-n}^{n} e^{ikt} = e^{-int} \sum_{k=0}^{2n} e^{ikt} = e^{-int} \frac{1-e^{(2n+1)it}}{1-e^{it}} \quad \text{(use induction)}$$

$$= \frac{e^{i(n+1/2)t} - e^{-i(n+1/2)t}}{e^{it/2} - e^{-it/2}} = \frac{\sin(n+\frac{1}{2})t}{\sin\frac{1}{2}t}$$

$\square$

Plugging this into $(*)$, we get (setting $t = \frac{a}{n}$):

$$S_n = \frac{a}{n}\left(\frac{\sin(n+\frac{1}{2})\frac{a}{n}}{2\sin\frac{a}{2n}} - \frac{1}{2} \right)$$

$$= \frac{\frac{a}{2n}}{\sin\frac{a}{2n}} \cdot \sin\left(a + \frac{a}{2n}\right) - \frac{a}{2n}.$$

As $\lim_{n\to\infty} \frac{\sin\frac{a}{2n}}{\frac{a}{2n}} = 1$, we get

$$\int_0^a \cos x \, dx = \lim_{n\to\infty} S_n = \sin a.$$

$\square$

<u>Proposition 7.8:</u>

Let $a < b < c$ and $f: [a,c] \to \mathbb{R}$ a function. f is integrable if and only if both $f|_{[a,b]}$ and $f|_{[b,c]}$ are integrable and we then have

$$\int_a^c f(x)\,dx = \int_a^b f(x)\,dx + \int_b^c f(x)\,dx.$$

Definition 7.6:

$$\int_a^a f(x)\,dx := 0,$$

$$\int_a^b f(x)\,dx := -\int_b^a f(x)\,dx, \quad \text{if } b < a.$$

§7.2 Indefinite Integral

Let $I \subset \mathbb{R}$ be an arbitrary interval and $a \in I$.

Proposition 7.9:

For $x \in I$ let

$$F(x) := \int_a^x f(t)\,dt. \qquad \text{"indefinite integral"}$$

Then $F : I \longrightarrow \mathbb{R}$ is differentiable and we have $F' = f$.

Proof:

For $h \neq 0$ we have

$$\frac{F(x+h) - F(x)}{h} = \frac{1}{h}\left(\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dx\right) = \frac{1}{h}\int_x^{x+h} f(t)\,dt.$$

According to the mean value theorem of integration (Prop. 7.7) there exists a $\xi_h \in [x, x+h]$ such that

$$\int_x^{x+h} f(t)\,dt = h f(\xi_h).$$

As $\lim_{h\to 0} \xi_h = x$ and f is continuous, we get

$$F'(x) = \lim_{h\to 0} \frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} \frac{1}{h}\left(h f(\xi_h)\right) = f(x).$$

$\square$

<u>Definition 7.7:</u>

A differentiable function $F: I \longrightarrow \mathbb{R}$ is called "primitive function" of a function $f: I \longrightarrow \mathbb{R}$, if $F' = f$. Thus the indefinite integral is a primitive function of the integrand.

<u>Proposition 7.10:</u>

Let $F: I \longrightarrow \mathbb{R}$ be an indefinite integral of $f: I \longrightarrow \mathbb{R}$. A second function $G: I \longrightarrow \mathbb{R}$ is then also an indefinite integral of f if and only if $F - G$ is a constant.

<u>Proof:</u>

i) Let $F - G = c$ with a constant $c \in \mathbb{R}$.
Then $G' = (F-c)' = F' = f$.

ii) Let G be indefinite integral of f, i.e. $G' = f = F'$.

Then $(F-G)' = 0 \Rightarrow F-G$ is constant. $\square$

Theorem 7.2 (Fundamental theorem of Calculus):
Let $f: I \longrightarrow \mathbb{R}$ be a continuous function and F an indefinite integral of f. Then we have for all $a,b \in I$:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof:
For $x \in I$ let

$$F_0(x) := \int_a^x f(t)\,dt.$$

Is F an arbitrary indefinite integral of f, then there exists according to Prop. 7.10 a $c \in \mathbb{R}$ with $F - F_0 = c$. Therefore,

$$F(b) - F(a) = F_0(b) - F_0(a) = F_0(b) = \int_a^b f(t)\,dt.$$

$\square$

Notation:
One sets $F(x)\Big|_a^b := F(b) - F(a)$.

$$\Rightarrow \int_a^b f(x)\,dx = F(x)\Big|_a^b.$$